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Appendix B
Sets and Relations

Your theory is crazy, but it's not crazy enough to be true.
—Niels Bohr, to a young physicist

First things first, but not necessarily in that order.
—Doctor Who

What we imagine is order is merely the prevailing form of chaos.
—Kerry Thornley, *Principia Discordia*, 5th edition

The art of progress is to preserve order amid change.
—A. N. Whitehead

Confusion is a word we have invented for an order which is not understood.
—Henry Miller (1891 – 1980)

Not till we are lost, in other words, not till we have lost the world, do we begin to find ourselves, and realize the infinite extent of our relations.
—Henry David Thoreau (1817 – 1862)

Throughout the book we have assumed a basic knowledge of set theory. This appendix provides a brief review of some of the basic concepts of set theory used in this book.

B.1 BASIC SET THEORY

A *set* S is any collection of objects that can be distinguished. Each object x which is in S is called a *member* of S (denoted $x \in S$). When an object x is not a member of S, it is denoted by $x \notin S$. A set is determined by its members. Therefore, two sets X and Y are equal when they consist of the same members (denoted $X = Y$). This means that if $X = Y$ and $a \in X$, then $a \in Y$. This is known as the *principle of extension.* If two sets are not equal, it is denoted $X \neq Y$. There are three basic properties of equality:

1. $X = X$ (*reflexive*)
2. $X = Y$ implies $Y = X$ (*symmetric*)
3. $X = Y$ and $Y = Z$ then $X = Z$ (*transitive*)

Example B.1.1 The set {1, 2, 3, 5, 6, 10, 15, 30} is the set whose members are the divisors of 30. The sets {30, 15, 10, 6, 5, 3, 2, 1} and {1, 1, 2, 3, 5, 5, 6, 10, 15, 30} are equal to the set {1, 2, 3, 5, 6, 10, 15, 30} since they have the same members.

Large or infinite sets are described using the help of *predicates.* A predicate $P(x)$ takes an object and returns true or false. When a set S is defined using a predicate $P(x)$, the set S contains those objects a such that $P(a)$ is true. This is known as the *principle of abstraction.* This is denoted using *set builder notation* as follows:

$$S = \{x \mid P(x)\}$$

This is read as "the set of all objects x such that $P(x)$ is true." There are some useful variants. For example, the following sets can be used interchangeably:

$$\begin{aligned} \{x \mid x \in A \text{ and } P(x)\} &= \{x \in A \mid P(x)\} \\ \{y \mid y = f(x) \text{ and } P(x)\} &= \{f(x) \mid P(x)\} \end{aligned}$$

Example B.1.2 The set $\{x \in \mathcal{N} \mid x \text{ divides } 30\}$, where $\mathcal{N}$ the set of natural numbers, is equivalent to the set $\{1, 2, 3, 5, 6, 10, 15, 30\}$.

Another useful relation on sets is *subset.* If X and Y are sets such that every member of X is also a member of Y, then X is a subset of Y (denoted $X \subseteq Y$). If, on the other hand, every member of Y is a member of X, then X is a superset of Y (denoted $X \supseteq Y$). If $X \subseteq Y$ and $X \neq Y$, then X is a proper subset of Y (denoted $X \subset Y$). Proper superset is similarly defined (denoted $X \supset Y$). The subset relation has the following three basic properties:

1. $X \subseteq X$ (*reflexive*)
2. $X \subseteq Y$ and $Y \subseteq X$ implies that $X = Y$ (*antisymmetric*)
3. $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$ (*transitive*)

The set which includes no elements is called the *empty set* (denoted $\emptyset$). For any set X, the empty set is a subset of it (i.e., $\emptyset \subseteq X$). Each set $X \neq \emptyset$ has at least two subsets X and $\emptyset$. Each member of a set $x \in X$ also determines a subset of X (i.e., $\{x\} \subseteq X$). Similarly, if a set has at least two members, each pair of objects makes up a subset. The *power set* of a set X is all subsets of X (denoted 2^X). The number of members of a set X is denoted $|X|$. The number of members of 2^X is equal to $2^{|X|}$.

Example B.1.3 If $X = \{2,3,5\}$ and $Y = \{1,2,3,5,6,10,15,30\}$, then

$$\begin{aligned} X &\subseteq Y \\ 2^X &= \{\emptyset, \{2\}, \{3\}, \{5\}, \{2,3\}, \{2,5\}, \{3,5\}, X\} \end{aligned}$$

The *union* of two sets X and Y (denoted $X \cup Y$) is the set composed of all objects that are a member of either X or Y (i.e., $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$). The *intersection* of two sets X and Y (denoted $X \cap Y$) is the set composed of all objects that are a member of both X and Y (i.e., $X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$).

Example B.1.4 If $X = \{2,3\}$ and $Y = \{2,5\}$, then

$$\begin{aligned} X \cup Y &= \{2,3,5\} \\ X \cap Y &= \{2\} \end{aligned}$$

Two sets X and Y are *disjoint* if their intersection contains no members (i.e., $X \cap Y = \emptyset$). Otherwise, the sets *intersect* (i.e., $X \cap Y \neq \emptyset$). A *disjoint collection* is a set of sets in which each pair of member sets is disjoint. A *partition* of a set X is a disjoint collection π of nonempty and disjoint subsets of X such that each member of X is contained within some set in π.

Example B.1.5 The set $\{\{1\}, \{2,3,5\}, \{6,10,15\}, \{30\}\}$ is a partition of the set $\{1,2,3,5,6,10,15,30\}$.

The set U is called the *universal set*, and it is composed of all objects in some domain being discussed. The *absolute complement* of a set X (denoted $\overline{X}$) are those elements in U which are not in X (i.e., $\{x \in U \mid x \notin X\}$). The *relative complement* of a set X with respect to a set Y (denoted $Y - X$) are those elements in Y which are not in X (i.e., $Y - X = Y \cap \overline{X} = \{x \in Y \mid x \notin X\}$). The *symmetric difference* of two sets X and Y (denoted $X + Y$) are those objects in exactly one of the two sets [i.e., $(A - B) \cup (B - A)$].

Example B.1.6 If $U = \{1,2,3,5,6,10,15,30\}$, $X = \{2,3\}$, and $Y = \{2,5\}$, then

$$\begin{aligned} \overline{X} &= \{1,5,6,10,15,30\} \\ X - Y &= \{3\} \\ X + Y &= \{3,5\} \end{aligned}$$

There are numerous useful *identities* (equations that are true regardless of what U and the subset letters represent) shown in Table B.1. Note that each

Table B.1 Useful identities of set theory.

Law	Union	Intersection
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
Communitive	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive	$A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A$	$A \cap U = A$
Inverse	$A \cup \overline{A} = U$	$A \cap \overline{A} = \emptyset$
Idempotent	$A \cup A = A$	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
DeMorgan	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$

identity appears twice and that the second can be found by interchanging $\cup$ and $\cap$ as well as $\emptyset$ and U. The identities in the third column are called *duals* of those in the second, and vice versa. In general, we can use this *principle of duality* to translate any theorem in terms of $\cup$, $\cap$, and complement to a dual theorem. One last useful property is the following equivalence:

$$A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$$

B.2 RELATIONS

Using set theory, we can define *binary relations* to show relationships between two items. Examples include things like "a is less than b" or "Chris is husband to Ching and father of John." First, we define an *ordered pair* as a set of two objects which have a specified order. An ordered pair of x and y is denoted by $\langle x, y \rangle$ and is equivalent to the set $\{\{x\}, \{x, y\}\}$. A binary relation is simply a set of ordered pairs. We say that x is *ρ-related to y* (denoted $x\rho y$) when ρ is a binary relation and $\langle x, y \rangle \in \rho$. The *domain* and *range* of ρ are

$$\begin{aligned} D_\rho &= \{x \mid \exists y \,.\, \langle x, y \rangle \in \rho\} \\ R_\rho &= \{y \mid \exists x \,.\, \langle x, y \rangle \in \rho\} \end{aligned}$$

Example B.2.1 A binary relation ρ that says that x times y equals 30 is defined as follows:

$$\rho = \{\langle 1, 30 \rangle, \langle 2, 15 \rangle, \langle 3, 10 \rangle, \langle 5, 6 \rangle, \langle 6, 5 \rangle, \langle 10, 3 \rangle, \langle 15, 2 \rangle, \langle 30, 1 \rangle\}$$

Using ordered pairs, we can recursively define an *ordered triple* $\langle x, y, z \rangle$ as being equivalent to the ordered pair $\langle \langle x, y \rangle, z \rangle$. A *ternary relation* is simply a set of ordered triples. We can further define for any size n an *ordered n-tuple* and use them to define *n-ary relations.*

One of the simplest binary relations is the *cartesian product*, which is the set of all pairs $\langle x, y\rangle$, where x is a member of some set X and y is a member of some set Y. It is defined formally as follows:

$$X \times Y \quad = \quad \{\langle x, y\rangle \mid x \in X \wedge y \in Y\}$$

If $X \supseteq D_\rho$ and $Y \supseteq R_\rho$, then $\rho \subseteq X \times Y$ and ρ is a *relation from X to Y*.

Example B.2.2 The cartesian product of $X = \{2,3,5\}$ and $Y = \{6,10\}$ is defined as follows:

$$X \times Y \quad = \quad \{\langle 2,6\rangle, \langle 2,10\rangle, \langle 3,6\rangle, \langle 3,10\rangle, \langle 5,6\rangle, \langle 5,10\rangle\}$$

A relation ρ in a set X is an *equivalence relation* iff it is reflexive (i.e., $x\rho x$ for all $x \in X$), symmetric (i.e., $x\rho y$ implies $y\rho x$), and transitive (i.e., $x\rho y$ and $y\rho z$ imply $x\rho z$). A set $A \subseteq X$ is an *equivalence class* iff there exists an $x \in A$ such that A is equal to the set of all y for which $x\rho y$. The equivalence class implied by x is denoted $[x]$. Using ρ, we can partition a set X into a set of equivalence classes called a *quotient set*, which is denoted by X/ρ.

Example B.2.3 The binary relation ρ on the set $X = \{1,2,3,5,6,10,15,30\}$ defined below is an equivalence relation.

$$\begin{aligned}
\rho \quad &= \quad \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,5\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,5\rangle, \langle 5,2\rangle, \langle 5,5\rangle, \\
&\qquad \langle 6,6\rangle, \langle 6,10\rangle, \langle 6,15\rangle, \langle 10,6\rangle, \langle 10,10\rangle, \langle 10,15\rangle, \langle 15,6\rangle, \\
&\qquad \langle 15,10\rangle, \langle 15,15\rangle, \langle 30,30\rangle\} \\
X/\rho \quad &= \quad \{\{1\}, \{2,3,5\}, \{6,10,15\}, \{30\}\}
\end{aligned}$$

A *function* is a binary relation in which no two members have the same first element. More formally, a binary relation f is a function if $\langle x, y\rangle$ and $\langle x, z\rangle$ are members of f, then $y = z$. If f is a function and $\langle x, y\rangle \in f$ (i.e., xfy), then x is an argument of f and y is the *image* of x under f. A function f is *into* Y if $R_f \subseteq Y$. A function f is *onto* Y if $R_f = Y$. A function f is *one-to-one* if $f(x) = f(y)$ implies that $x = y$. Functions can be extended to more variables by using arguments that are ordered n-tuples.

Example B.2.4 The function f on the set $X = \{1,2,3,5,6,10,15,30\}$ is defined as the result of dividing 30 by x. It is onto X and one-to-one.

$$f \quad = \quad \{\langle 1,30\rangle, \langle 2,15\rangle, \langle 3,10\rangle, \langle 5,6\rangle, \langle 6,5\rangle, \langle 10,3\rangle, \langle 15,2\rangle, \langle 30,1\rangle\}$$

A binary relation ρ is called a *partial order* if it is reflexive, antisymmetric (i.e., $x\rho y$ and $y\rho x$ implies that $x = y$), and transitive. A *partially ordered set (poset)* is a pair $\langle X, \leq\rangle$, where $\leq$ partially orders X. A partial order is a *simple* (or *linear*) *ordering* if for every pair of elements from the domain x and y either $x\rho y$ or $y\rho x$. An example of a simple ordering is $\leq$ on the real numbers. A *simply ordered set* is also called a *chain.* Two posets $\langle X, \leq\rangle$ and $\langle X', \leq'\rangle$ are isomorphic if there exists a one-to-one mapping between X and X' that preserves the ordering.

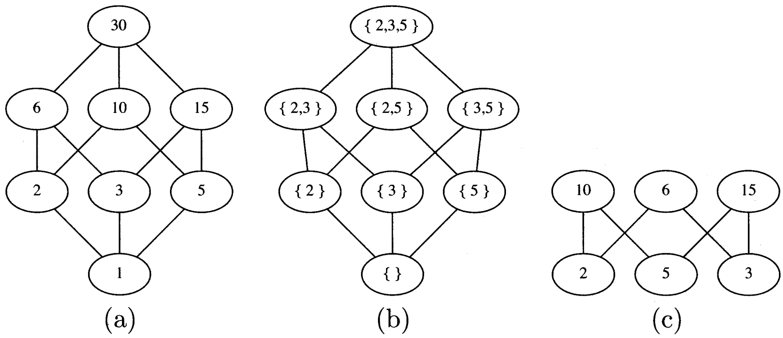

Fig. B.1 (a) Poset composed of the divisors of 30. (b) Isomorphic poset for the power set of {2, 3, 5}. (c) Poset with no least or greatest member.

Example B.2.5 Consider the set $\{1, 2, 3, 5, 6, 10, 15, 30\}$. It can be partially ordered using the relation $\leq$, which is defined to be x divides y. A diagram for this poset is shown in Figure B.1(a), where y is placed above x if $x \leq y$. Reflexive and transitive edges are omitted. A poset created from the power set of $\{2, 3, 5\}$ with $\subseteq$ as the ordering relation is shown in Figure B.1(b). These two posets are isomorphic.

A *least member* of X with respect to $\leq$ is a x in X such that $x \leq y$ for all y in X. A least member is unique. A *minimal member* is a x in X such that there does not exist a y in X such that $y < x$. A minimal member need not be unique. Similarly, a *greatest member* is a x in X such that $y \leq x$ for all y in X. A *maximal member* is a x in X such that there does not exist a y in X such that $y > x$. A poset $\langle X, \leq \rangle$ is *well-ordered* when each nonempty subset of X has a least member. Any well-ordered set must be a chain.

Example B.2.6 The least and minimal member of the poset shown in Figure B.1(a) is 1. The greatest and maximal member is 30. The poset shown in Figure B.1(c) has no least or greatest member. Its minimal members are 2, 3, and 5. Its maximal members are 6, 10, and 15.

For a poset $\langle X, \leq \rangle$ and $A \subseteq X$, an element $x \in X$ is an *upper bound* for A if for all $a \in A$, $a \leq x$. It is a *least upper bound* for A [denoted $lub(A)$] if x is an upper bound and $x \leq y$ for all y which are upper bounds of A. Similarly, an element $x \in X$ is a *lower bound* for A if for all $a \in A$, $x \leq a$. It is a *greatest lower bound* for A [denoted $glb(A)$] if x is a lower bound and $y \leq x$ for all y which are lower bounds of A. If A has a least upper bound, it is unique, and similarly for the greatest lower bound.

Example B.2.7 Consider the poset shown in Figure B.1(a). The upper bounds for $A = \{2, 5\}$ are 10 and 30, while lub(2,5) is 10. The lower bounds for $A = \{6, 15\}$ are 1 and 3, while the glb(6,15) is 3. For the poset shown in Figure B.1(c), the lub(3,10) does not exist.